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Uniformly bounded Riesz bases and equiconvergence theorems

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ABSTRACT: We review some classical results on the convergence of classical trigonometric and polynomial Fourier series. Then we present a not well-known short proof of the local uniform boundedness of many classical orthonormal systems. Finally, we formulate a strong generalization of Haar's classical equiconvergence theorem.

Key Words: Fourier Series, Riesz basis, classical orthogonal polynomials

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Contents

1. Review of Fourier series

In this section we recall some classical results on the convergence of Fourier series. See [6] and [11] for references.

Fourier series in Hilbert spaces. We recall that if (u_k) is an orthonormal basis in a Hilbert space H, then every function $f \in H$ may be developed into a convergent Fourier series

$$f = \sum (f, u_k) u_k,$$

the convergence taking place in norm.

Convergence of trigonometric Fourier series in L^p norms. Consider the special case of trigonometric Fourier series in the Hilbert space $H = L^2(-\pi, \pi)$. As a special case of the above result, the trigonometric Fourier series of every $f \in L^2(-\pi, \pi)$ converges to f in $L^2(-\pi, \pi)$.

Since the trigonometric system is uniformly bounded, we may define the Fourier series of every $f \in L^1(-\pi, \pi)$ and so we may investigate the norm convergence in

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the spaces $L^p(-\pi,\pi)$, $1 \le p \le \infty$. There are some surprises. We recall the following results:

- If $f \in L^1$, then its Fourier series may diverge in L^1 norm (Hahn, 1916).
- Similarly, if $f \in L^\infty$, then its Fourier series does not converge necessarily in L^∞ norm. Both results may be proven conveniently today by estimating the Lebesgue constants and applying the Banach–Steinhaus theorem.
- On the other hand, if $f \in L^p$ for some $1 < p < \infty$, then its Fourier series converges to f in the L^p norm. This was proved by M. Riesz in 1927 using conjugate Fourier series.

Pointwise convergence of trigonometric Fourier series. A famous example of Du Bois Reymond (1873) showed that the trigonometric Fourier series of a continuous function may diverge at some points. Lusin asked in 1913 whether the trigonometric Fourier series of a continuous function converges at least almost everywhere. The main results are as follows:

- if the trigonometric Fourier series of a function $f \in L^1$ converges almost everywhere to some function g, then $f = g$ almost everywhere (Fejér, 1900, and Lebesgue, 1905).
- the Fourier series of certain L^1 functions *diverges* everywhere (Kolmogorov, 1926).
- the Fourier series of every L^p function with $p > 1$ converges almost everywhere (Carleson, (1966) for $p = 2$, Hunt (1968) for the general case). Carleson's theorem answered affirmatively Lusin's question.

Remark 1.1

- *The fact that the trigonometric Fourier series of an L^2 function can only converge to f almost everywhere, remains valid for every orthonormal basis in any L^2 space. Indeed, the series converges in norm and then a suitable subsequence of the partial sums converges to f almost everywhere by a lemma of Riesz (what he used for a simple proof of the Riesz–Fischer theorem).*
- *Banach's first publication (in collaboration with Steinhaus in 1919) points out the subtlety of the relation between norm convergence and pointwise convergence: they constructed a function $f \in L^1$ whose Fourier series converges almost everywhere but diverges in the L^1 norm.*

Pointwise convergence of non trigonometric Fourier series. Let us recall an intriguing example of Banach (1923): there exist $f, g \in L^1(-\pi,\pi)$ and an orthonormal basis in $H = L^2(-\pi,\pi)$ formed by bounded functions such that $f \ne g$ everywhere but the Fourier series of f converges to g everywhere.

This phenomenon does not occur for trigonometric Fourier series and for most classical orthonormal expansions (Legendre, Jacobi, Laguerre, Hermite). As we shall see, this is related to the fact these orthonormal bases are formed of eigenfunctions of special differential operators.

2. Uniform boundedness of eigenfunctions

Let (a, b) be a bounded open interval and $q \in L^1(a, b)$. The following theorem[1] is due to V. A. Il'in and I. Joó (1979):

Theorem 2.1 *If $u \in W^{2,1}(\Omega)$ and*

$$-u'' + qu = \lambda u \quad \text{in} \quad (a, b)$$

for some $\lambda \geq 0$, then

$$\|u\|_\infty \leq C\|u\|_2$$

with C depending only on the length of (a, b) and on $\|q\|_1$.

We are going a somewhat simplified proof here. We need a mean-value formula:

Lemma 2.2 *If $x \pm t \in (a, b)$, then*

$$u(x-t) + u(x+t) - 2u(x)\cos\sqrt{\lambda}t = \int_{x-t}^{x+t} q(s)u(s)\frac{\sin\sqrt{\lambda}(t - |x-s|)}{\sqrt{\lambda}}\ ds.$$

Proof: Since $qu = u'' + \lambda u$, integrating by parts we obtain

$$\int_{x-t}^{x} q(s)u(s)\frac{\sin\sqrt{\lambda}(t-|x-s|)}{\sqrt{\lambda}}\ ds = u'(x)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} - u(x)\cos\sqrt{\lambda}t + u(x-t)$$

and

$$\int_{x}^{x+t} q(s)u(s)\frac{\sin\sqrt{\lambda}(t-|x-s|)}{\sqrt{\lambda}}\ ds = -u'(x)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} - u(x)\cos\sqrt{\lambda}t + u(x+t);$$

adding them we obtain the required formula. □

Proof of Theorem 2.1: Fix $R > 0$ such that $4R \leq b - a$ and $R\|q\|_1 \leq 1$. If x belongs to the left half of (a, b), then we infer from the lemma the following inequality:

$$|u(x)| \leq |u(x+2t)| + 2|u(x+t)| + |t| \cdot \|q\|_1\|u\|_\infty.$$

Integrating this inequality for $0 < t < R$ we get

$$\begin{aligned} R|u(x)| &\leq \int_0^R |u(x+2t)|\ dt + 2\int_0^R |u(x+t)|\ dt + \frac{R^2}{2}\|q\|_1\|u\|_\infty \\ &\leq 3\sqrt{R}\|u\|_2 + \frac{R}{2}\|u\|_\infty. \end{aligned}$$

[1] Here and in the sequel we denote by $\|\cdot\|_p$ the norm of $L^p(a, b)$.

The same estimate holds for the right half of (a, b), too, so that

$$R\|u\|_\infty \le 3\sqrt{R}\|u\|_2 + \frac{R}{2}\|u\|_\infty$$

and

$$\|u\|_\infty \le \frac{6}{\sqrt{R}}\|u\|_2.$$

□

Corollary 2.3 *If (u_k) is a bounded sequence in $L^2(a, b)$ and*

$$-u_k'' + qu_k = \lambda_k u_k \quad in \quad (a, b), \quad k = 1, 2, \dots$$

with some $\lambda_k \ge 0$ where $q \in L^1(a, b)$, then (u_k) is uniformly bounded.

Example 2.4 *The trigonometric system is uniformly bounded.*

We apply the corollary to the Jacobi, Laguerre and Hermite polynomials.

Proposition 2.5

(a) The orthonormal Jacobi polynomials $P_k^{(\alpha,\beta)}$ are uniformly bounded on the compact subintervals of $(-1, 1)$.

(b) The orthonormal Laguerre polynomials $L_k^{(\alpha)}$ are uniformly bounded on the compact subintervals of $(0, \infty)$.

(c) The orthonormal Hermite polynomials H_k are uniformly bounded on the compact subintervals of $\mathbb{R}$.

Proof: [2]

(a) Writing

$$\begin{aligned}
1 &= \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta |P_k^{(\alpha,\beta)}(x)|^2 \, dx \\
&= 2^{\alpha+\beta+1} \int_0^\pi \left(\sin\frac{\theta}{2}\right)^{2\alpha+1} \left(\cos\frac{\theta}{2}\right)^{2\beta+1} |P_k^{(\alpha,\beta)}(\cos\theta)|^2 \, d\theta \\
&=: 2^{\alpha+\beta+1} \int_0^\pi |u_k(\theta)|^2 \, d\theta
\end{aligned}$$

and

$$q(\theta) := \frac{4\alpha^2 - 1}{16\sin^2\frac{\theta}{2}} + \frac{4\beta^2 - 1}{16\cos^2\frac{\theta}{2}}$$

we have (see [10], p. 67)

$$-u_k'' + qu_k = \left(k + \frac{\alpha+\beta+1}{2}\right)^2 u_k.$$

[2] The usual proofs, based on complex analysis, are substantially longer.

We conclude by applying the corollary for any interval (a,b) satisfying $[a,b] \subset (-1,1)$.

(b) Putting

$$u_k(x) := e^{-x^2/2} x^{\alpha+\frac{1}{2}} L_k^{(\alpha)}(x^2)$$

and

$$q(x) := x^2 + \frac{4\alpha^2 - 1}{4x^2}$$

we have (see [10], p. 100)

$$-u_k'' + qu_k = (4k + 2\alpha + 2)u_k.$$

We conclude by applying the corollary for any interval (a,b) satisfying $[a,b] \subset (0,\infty)$.

(c) Putting

$$u_k(x) := e^{-x^2/2} H_k(x)$$

and

$$q(x) := x^2$$

we have (see [10], p. 106)

$$-u_k'' + qu_k = (2k+1)u_k.$$

We conclude by applying the corollary for any bounded interval (a,b). □

We end this section with two remarks. First, Theorem 2.1 may be generalized to complex eigenvalues and the resulting estimates are optimal (see [4] and [7]):

Theorem 2.6 *Let (a,b) be an open interval and $q \in L^1(a,b)$. If $u \in W^{2,1}(a,b)$ and*

$$-u'' + qu = \lambda u \quad \text{in} \quad (a,b)$$

for some $\lambda \in \mathbb{C}$, then

$$c_1(1 + |\Im\sqrt{\lambda}|)^{\frac{1}{p}-\frac{1}{q}} \le \frac{\|u\|_q}{\|u\|_p} \le c_2(1 + |\Im\sqrt{\lambda}|)^{\frac{1}{p}-\frac{1}{q}}$$

for all $p, q \in [1,\infty]$ with c_1, c_2 depending only on the length of (a,b) and on $\|q\|_1$.

On the other hand, the situation is different in higher dimension:

Example 2.7 *Consider an orthonormal basis (u_k) of eigenfunctions of $-\Delta$ in a three-dimensional ball Ω of radius π with homogeneous Dirichlet boundary conditions. Then (u_k) is not uniformly bounded.*[3] *Indeed, the formula*

$$u_k(x) := \frac{\sin k|x|}{\sqrt{2}\pi|x|}, \qquad k = 1, 2, \ldots$$

[3] The same result holds for all balls in all dimensions ≥ 2 and also for other types of boundary conditions.

defines a subsequence of the orthonormal basis of eigenfunctions, and we have

$$u_k(0) = \frac{k}{\sqrt{2\pi}} \to \infty.$$

3. Distribution of eigenvalues

The above mean-value formula also allows us to prove in an elementary way that the eigenvalues of many classical orthonormal systems have no finite accumulation points. In this section we consider a bounded interval (a, b), an integrable function $q \in L^1(a, b)$ and an orthonormal basis (u_k) in $L^2(a, b)$ such that each u_k belongs to $W^{2,1}(a, b)$ and satisfies the differential equation

$$-u_k'' + qu_k = \lambda_k u_k \quad \text{in} \quad (a, b)$$

with a suitable real number λ_k. The following result was obtained in [8].

Theorem 3.1 *We have $\lambda_k \to \infty$.*

Remark 3.2

- *Usually results of this type are obtained for special orthonormal systems whose elements satisfy specific boundary conditions. Let us emphasize that no boundary conditions are imposed here.*
- *The result and its proof remains valid if (u_k) is not an orthonormal basis but only a Bessel system, i.e.,*

  $$\sum |(f, u_k)|^2 < \infty$$

  for every $f \in L^2(a, b)$.

Sketch of proof: Fix $\mu \in \mathbb{C}$ arbitrarily and consider $u = u_k$ and $\lambda = \lambda_k$ with $|\mu - \sqrt{\lambda_k}| \le 1$. For x in the left half of (a, b) we integrate for $0 < t < R$ the mean value formula in the form

$$\begin{aligned} u(x) = {} & 2u(x+t)\cos\mu t - u(x+2t) \\ & + 2u(x+t)(\cos\sqrt{\lambda}t - \cos\mu t) \\ & + \int_x^{x+2t} q(s)u(s)\frac{\sin\sqrt{\lambda}(t - |x+t-s|)}{\sqrt{\lambda}}\ ds. \end{aligned}$$

We get

$$\begin{aligned} R|u(x)| &\le \int_a^b f_x u\ ds + c_1 \int_0^R t|u(x+t)|\ dt + R^2\|q\|_1\|u\|_\infty \\ &\le \int_a^b f_x u\ ds + c_2 R^2 \|u\|_\infty \\ &\le \int_a^b f_x u\ ds + c_3 R^2 \|u\|_2. \end{aligned}$$

Hence

$$R^2|u(x)|^2 \le 2(f_x,u)^2 + 2c_3^2R^4\|u\|_2^2$$

and by Bessel's inequality,

$$R^2 \sum_{|\mu-\sqrt{\lambda_k}|\le 1} |u_k(x)|^2 \le 2\|f_x\|_2^2 + c_4R^4 \sum_{|\mu-\sqrt{\lambda_k}|\le 1} \|u_k\|_2^2.$$

A similar estimate holds in the right half of (a,b). Integrating we get

$$R^2 \sum_{|\mu-\sqrt{\lambda_k}|\le 1} 1 \le 2\int_a^b \|f_x\|_2^2\ dx + c_4(b-a)R^4 \sum_{|\mu-\sqrt{\lambda_k}|\le 1} 1.$$

Choosing a small $R>0$ we conclude that

$$\sum_{|\mu-\sqrt{\lambda_k}|\le 1} 1 \le \frac{4}{R^2}\int_a^b \|f_x\|_2^2\ dx < \infty.$$

□

4. Equiconvergence

If we develop a function into Fourier series with respect to two different orthonormal bases, it is natural to ask how different the resulting series can be. The case of Sturm–Liouville type orthonormal bases was investigated by Haar in 1910 and 1918. The boundary conditions played an important role. Much more general theorems can be obtained by eigenfunction considerations.

As before, let (a,b) be a bounded interval, $q \in L^1(a,b)$ and (u_k) an orthonormal basis in $L^2(a,b)$ such that each u_k belongs to $W^{2,1}(a,b)$ and satisfies

$$-u_k'' + qu_k = \lambda_k u_k \quad \text{in} \quad (a,b)$$

with a suitable complex number λ_k.

For $f \in L^2(a,b)$ and $\mu > 0$ we set

$$\sigma_\mu(f) := \sum_{|\Re\sqrt{\lambda_k}|<\mu} (f,u_k)u_k$$

and

$$S_\mu(f,x) := \int_{x-R}^{x+R} \frac{\sin\mu(x-y)}{\pi(x-y)} f(y)\ dy \quad \text{whenever} \quad x \pm R \in (a,b).$$

Then the following results holds:

Theorem 4.1 *Given $[a',b'] \subset (a,b)$ fix $R>0$ such that $(a'-R,b'+R) \subset (a,b)$. Then*

$$\sigma_\mu f - S_\mu f \to 0 \quad \textit{uniformly in } (a',b') \textit{ if } \quad \mu \to \infty$$

for every $f \in L^2(a,b)$.

Example 4.2 *Carleson's theorem also holds for Jacobi polynomial expansions.*

To end this review we note that the above theorems remains valid

- for Riesz bases instead of orthonormal bases;
- for higher-order eigenfunctions;
- for more general differential operators

$$Lu := u^{(n)} + q_2 u^{(n-2)} + \cdots + q_n u, \quad q_s \in H^{n-s}_{\text{loc}}(a, b).$$

We refer to [9] and its references for details.

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